Polynomial-time Approximation Scheme for Minimum k-way Cut in Planar and Minor-free Graphs

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Abstract

The k-cut problem asks, given a graph G and a positive integer k, to find a minimum-weight set of edges whose removal splits G into at least k components. We give the first polynomial-time algorithm with approximation factor $2-\epsilon$ (with constant $\epsilon > 0$) for the k-cut problem on planar and minor-free graphs. Applying more complex techniques, we further improve our method and present a polynomial-time approximation scheme in both planar and minor-free graphs. Despite persistent effort, to the best of our knowledge, this is the first improvement for the k-cut problem over standard approximation factor of 2 in any major class of graphs.

1 Introduction

In the *k-cut problem*, given an undirected graph with edge weights, the goal is to find a minimum-weight set of edges whose removal splits the graph into at least k components. The problem is sometimes also called the *k-way cut problem* or the *multi-component cut*. This problem is a natural generalization of the minimum cut problem in which we want to find a minimum-weight set of edges whose removal splits the graph into two components.

Goldschmidt and Hochbaum [13] proved that the k-cut problem is NP-hard when k is part of the input. In the same work, they provided an $O(n^{k^2})$ algorithm for the k-cut problem which is polynomial for every fixed k. Better algorithms have been proposed in a series of works [15, 29, 17, 28]. As of today, the best algorithm for the minimum k-cut problem is by Thorup [28], and has the running time of $O(n^{2k} \log n)$. Despite these improvements, this problem is proven to be W[1]-hard when k is taken as a parameter [10] which shows that this problem does not have a FPT algorithm unless $\mathsf{P} = \mathsf{NP}$.

In terms of approximation algorithms, several approximation algorithms are known for this problem [26, 21, 24, 32, 30], however the approximation ratio of none of them is better than $2 - o(1)$. In fact, a very recent result by Manurangsi [20] shows that this problem is NP-hard to approximate to within $2-\epsilon$ factor assuming Small Set Expansion Hypothesis. Additionally, to the best of our knowledge, prior to this work, there was no approximation algorithm with a ratio better than 2 for any major class of graphs. It is also worth mentioning that a recent work by Gupta et al. [14] showed that using an FPT algorithm the approximation factor of 2 can be beaten in the k-cut problem. They showed that there exists a $2-\epsilon$ approximation algorithm that runs in time $2^{O(k^6)}\tilde{O}(n^4)$.

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In this paper, we first show that surprisingly the approximation guarantee of a natural greedy algorithm is $2 - \epsilon$ in planar graphs as well as graphs excluding a fixed minor, for some positive constant ϵ. Later, we show how our method can be extended to derive a PTAS for the k-cut problem in minor-free graphs. This is the first result that beats the approximation factor of 2 in a polynomial-time for a major class of graphs.

Theorem 1.1. *The approximation ratio of the greedy algorithm presented as Algorithm 1 is* 1.9988 *for planar graphs and* $2 - \epsilon$ *for* H*-minor-free graphs where* $\epsilon > 0$ *is a constant depending on* H*.*

The above theorem is proved in Section 3. Then we move on to the result with the better guarantee. The following is proved in Section 4.

Theorem 1.2. *There is a polynomial-time approximation scheme (*PTAS*) for the* k*-cut problem in minor-free graphs.*

We work with the notion of the "density of cuts" which is the weight of a cut divided by its separation degree. Although the density of a minimum cut could be twice the density of the optimal solution, we show that the density of cuts with a larger separation degree converges to the density of the optimal solution in planar and minor-free graphs. Interestingly, the same does not hold in general graphs, where the density of arbitrary large cuts may be as much as a factor $2 - o(1)$ of the density of the optimal solution.

We show that a natural greedy algorithm that repeatedly picks a minimum-density split with a constant separation degree achieves an approximation ratio better than 2. First, in order to introduce and highlight our main ideas, we consider the greedy algorithm which repeatedly picks a minimum-density split with a separation degree of at most 3 and show that its approximation ratio is $2 - \epsilon$ in minor-free graphs. Subsequently, we generalize our method to derive a polynomial-time approximation scheme (PTAS) in minor-free graphs. Saran and Vazirani [26] considers a similar greedy algorithm which successively removes the edges of a minimum cut. They showed that the approximation ratio of the greedy algorithm is $2 - 2/k$ in general graphs. Later, Xiao et al. [30] generalized this method by repeatedly removing the edges of a minimum h-way split. Although they find a larger split in each iteration, they proved that the approximation ratio of this algorithm is about $2 - h/k$, and it does not beat the approximation factor of 2 by any constant factor.

In our first main result, we show that the approximation ratio of the simple greedy algorithm is better than 2 in minor-free graphs. Although the analysis of our second main result shows the existence of a PTAS algorithm in minor-free graph, it doesn't beat the approximation factor of 2 using the splits with a very small separation degree. Therefore, we provide a slightly different analysis for our first result. Our main observation is that in a balanced weighted graph, there is a matching that its weight is at least a constant fraction of the total weight of the graph. This result can also be viewed as a generalization of the work of Nishizeki and Baybars [22] in unweighted graphs. Later, we introduce a more profound analysis of our method to derive a PTAS in planar and minor-free graphs.

1.1 Related Works

A problem closely related to k-cut is the multiway cut problem. Given a set of k vertices called *terminals*, in the multiway cut problem, we want to find a minimum-weight cut that separates the terminals from one another. The study of its computational complexity was inaugurated in 1983 by Dahlhaus, Johnson, Papadimitriou, Seymour, and Yannakakis [9][1]. They provided a simple 2-approximation algorithm for the multiway cut problem, and proved that the problem is APX-hard

[1] The work was first known in an unpublished but widely circulated extended abstract. Their complete paper was published in 1994.

for any fixed $k \geq 3$. However, in the case of planar graphs, they showed that the problem can be solved in polynomial time for fixed k but is NP-hard when k is unbounded.

The approximation factor of this problem improved in a sequence of works [5, 8, 16]. As of today, the best approximation factor is 1.3438 [16]. In case of planar graphs, a very recent result by Bateni et al. [2] shows that there is a PTAS for the multiway cut problem in planar graphs.

Another problem related to the k-cut problem is the *Steiner k-cut problem*, which generalizes both the k-cut problem and the multiway cut problem. Given an edge-weighted undirected graph G, a subset of vertices T called terminals, and an integer $k \leq |T|$, the objective is to find a minimum-weight set of edges whose removal results in k disconnected components, each containing at least one terminal. The best result known for this problem is a $2 - 2/k$ approximation algorithm due to Chekuri et al. [6].

We remark that the "identity-relaxed" variants of Steiner tree and multiway cut problems, namely k-MST and k-cut, have been elusive to date. The latter problems allow us to pick the identity of k "terminals" to connect or separate, respectively. The initial 2-approximation algorithms for Steiner tree [12] and multiway cut [9] were improved in a series of work [5, 18, 23, 25, 31] culminating in a 1.3863 approximation algorithm [4] for Steiner tree and a 1.3438 approximation algorithm for multiway cut [16]. Nonetheless, no approximation guarantee better than $2 - o(1)$ is known for k-MST or k-cut.

Similarly, in the case of planar graphs, where PTASes are known for Steiner tree [3] and multiway cut [2], their identity-relaxed variants (prior to this work) proved to be more resilient. In particular, the standard spanner construction techniques and the small-treewidth reduction approach developed and successfully applied to a host of network design problems in the last decade, seem challenging to use in this context. Recently, Cohen-Addad et al. [7] gave PTASes for k-means and k-median, using the *local search method*. (In their case, the non-identity-relaxed variant where the k "centers" are known is trivial and not interesting to solve.)

2 Preliminaries

Let $G = (V, E; w)$ be an undirected graph where $w : E \rightarrow R^+$ is an assignment of weights to the edges of G. We use $V(G)$ and $E(G)$ to denote the vertices and edges of the graph G respectively. For each edge $e \in E$, we use $w(e)$ to denote the weight of e. Similarly, for a set of edges $E' \subseteq E$, we use $w(E')$ to denote the total weight of the edges in E', i.e., $w(E') = \sum_{e \in E'} w(e)$. A graph G is called *normalized* if $w(E) = 1$. We denote the number of (connected) components in G by $comp(G)$. Moreover, we use $\beta(G) = |E|/|V|$ to denote the ratio of the number of edges in G to its number of vertices.

A k-way cut is a partition of V into k disjoint, nonempty sets $V_1, V_2, \ldots, V_k$, called *parts*. We use $(V_1, V_2, \ldots, V_k)$ to denote the cut. The weight of a k-way cut is the total weight of the edges whose endpoints are in different parts. We denote the weight of the cut by $w(V_1, V_2, \ldots, V_k)$. For any k-way cut, we define its *separation degree* to be k.

For any subset $S \subseteq E$ of edges, we use $G - S$ to denote the graph derived from G by removing the edges in S. We say that the edge set S is a k-way split in G if $comp(G - S) = (k-1) + comp(G)$. Therefore, k-way splits are k-way cuts equivalent in connected graphs. Similarly, we define *separation degree* of S to be k. We use $d_G(S)$ to denote the *density* of S and define it as,

$$d_G(S) = w(S)/(k-1) .$$

A graph G is called H-minor free if and only if the graph H does not appear as a minor of G; i.e., H cannot be obtained via removing and contracting edges in G.

Algorithm 1: $2-\epsilon$ Approximation Algorithm for Minor-free Graphs

Data: An H-minor-free connected graph G, and integer k

$C = \emptyset$.
while separation degree of C is less than k **do**
  Let $G' = G - C$ be the graph obtained by removing all the previous cuts from G.
  Let d be the separation degree of C.
  if $k - d \leq 3$ **then**
    Let C' be a minimum $(k-d+1)$-way split in G'.
  else
    Let C' be a split which its density is minimum among all splits with the separation degree at most 3.
  $C = C \cup C'$.
return C.

For an H-minor-free graph G, we use $\eta(G)$ to denote the Hadwiger number of G which is the size of the largest complete graph that is a minor of G. The result by Thomason [27] directly implies that the number of edges in a H-minor-free graph with is linear in its number of edges.

Lemma 2.1. *For any H-minor-free graph G, we have $\beta(G) \leq (\gamma + o(1))|V(H)|\sqrt{\ln |V(H)|}$, where $\gamma = 0.319...$ is an explicit constant.*

Proof. Consider a complete graph H' with $|V(H)|$ vertices. This graph has H as its minor, thus G does not have a minor H'. Therefore $\eta(G) < |V(H)|$. It is shown in [27] that for every graph G we have

$$\beta(G) \leq (\gamma + o(1))(\eta(G)+1)\sqrt{\ln(\eta(G)+1)}\,,$$

where $\gamma = 0.319...$ is an explicit constant. This readily gives

$$\beta(G) \leq (\gamma + o(1))|V(H)|\sqrt{\ln |V(H)|}\,.$$

□

3 Beating Approximation Factor of 2 in Minor-free Graphs

In this section, we provide a $2-\epsilon$ approximation algorithm for the k-way cut problem in minor-free graphs. We repeatedly find minimum 3-way and 2-way splits, and pick the one with the lower density. Then, we remove the edges of this split from our graph to increase its number of connected components. The only exception is the last split picked by the algorithm in which its separation degree could be as large as 4. We show that in minor-free graphs, the approximation ratio of the same algorithm is better than 2 by a constant factor. However, we show that in minor-free graphs, the approximation ratio of our algorithm is better than 2 by a constant factor.

First, we show that the density of a split which has the minimum density among all the splits with the separation degree of at most 3, is at most $(2-\epsilon)OPT/k$ where OPT is the cost of the minimum k-way cut, and ϵ is a positive constant depending on $|V(H)|$. Later we use this theorem to show that the approximation ratio of our algorithm is better than 2. Our main tool is the following lemma which shows that in every balanced minor-free graph, there exists a matching that its weight is at least a constant fraction of the total weight of the graph. last split

Lemma 3.1. *Given an $\delta > 0$, let $G = (V, E; w)$ be a normalized H-minor-free graph with n vertices such that the density of 2-way split is least $(1+\delta)/n$, then the weight of a maximum weighted matching in G is at least $\dfrac{\delta^2}{16\beta(G)(1+\delta)}$.*

Proof. Let A be set of vertices in G whose degree is at least d. It follows that $|A| \leq 2|E|/d$. Note that $|E| = n\beta(G)$. Therefore, $|A| \leq 2n\beta(G)/d$. Let $B = V \setminus A$, then $|B| \geq n(1 - 2\beta(G)/d)$. Let E_B be the set of edges in E whose both ends are in B, and E_A be all other edges. Setting $d = 4\beta(G)(1+\delta)/\delta$, we claim that $w(E_B) \geq \delta/2$.

For the sake of contradiction suppose that $w(E_B) < \delta/2$. We have

$$w(E_A) = w(E) - w(E_B) > w(E) - \delta/2\,.$$

Since G is normalized, we have $w(E) = 1$. Therefore,

$$w(E_A) > 1 - \delta/2\,.$$

For every $u \in B$, let C_u be the 2-way split in G which separates u from all other vertices. Considering all C_u splits, each edge in E_B appears in 2 of these splits, and each edge in E_A appears in at most one of them. Thus,

$$\sum_{u\in B} w(C_u) \leq 2w(E_B) + w(E_A)\,.$$

Note that $w(E_B) + w(E_A) = w(E) = 1$. Therefore, we have

$$\sum_{u\in B} w(C_u) \leq 2w(E_B) + w(E_A) < 1 + \delta/2\,. \tag{1}$$

On the other hand, since every cut C_u is a 2-way split and has the density at least $(1+\delta)/n$, we have

$$\sum_{u\in B} w(C_u) > \frac{1+\delta}{n}|B| \geq (1+\delta)(1 - 2\beta(G)/d)\,.$$

Substitute d for $4\beta(G)(1+\delta)/\delta$, gives us,

$$\sum_{u\in B} w(C_u) \geq 1 + \delta/2\,. \tag{2}$$

Inequality (1) together with (2) is a contradiction. Therefore, $w(E_B) \geq \delta/2$.

Now we find a weighted matching using the following greedy algorithm.

1. Let $T = E_B$ be the set all the edges in E_B, and $\mathcal{M} = \emptyset$ be our current matching.

2. Let $e \in T$ be the edge which has the maximum weight among all the edges in T.

3. Add e to the matching, $\mathcal{M} = \mathcal{M} \cup \{e\}$. Also, remove all the edges which are incident to e from T.

4. While $|T| > 0$, repeat steps 2-3.

In each step, we pick a edge which has the maximum weight in T, add it to our current matching, and remove all the edges which are incident to this edge from T. Since, degree of every vertex in B is at most d, every time we add an edge to our matching, we remove at most $2d-1$ edges from T. Let e be the edge picked by Algorithm in one of its steps. e has the maximum weight in T, thus the weight of each of the removed edges in this step is at most $w(e)$. Therefore,

$$w(\mathcal{M}) \geq \frac{w(E_B)}{2d-1} \geq \frac{\delta/2}{2d}\,.$$

Replacing d, we have

$$w(\mathcal{M}) \geq \frac{\delta/2}{2d} = \frac{\delta^2}{16\beta(G)(1+\delta)}\,.$$

Therefore, we have found a matching with the total weight of $\frac{\delta^2}{16\beta(G)(1+\delta)}$, and it completes the proof. Note that by Lemma 2.1, $\beta(G)$ is at most a constant, therefore we have found a matching with a constant weight in G. □

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Theorem 3.2. *Given an H-minor-free graph G and an integer $k \geq 3$, let S be a split with the minimum density among all the splits with the separation degree of at most 3, Then for any k-way split S_k, we have*

$$d_G(S) \leq \frac{(2-\epsilon)w(S_k)}{k}\,,$$

where $\epsilon > 0$ is a constant depending on $|V(H)|$.

Proof. First consider the case that G is connected. Let $P_1, P_2, \ldots, P_k$ be the components in $G - S_k$. For each P_i let E_i be set of edges whose both ends are in P_i. We contract all the edges in $E_1, E_2, \ldots, E_k$ to obtain the new graph $G' = (V', E'; w')$. Also, we replace parallel edges with a single edge with the weight of sum of them. The graph G' has exactly k vertices each corresponding to a component in $G - S_k$. Furthermore, G' is H-minor-free since it is derived by edge contradictions from G. Moreover, every split in G' corresponds to a split with the same separation degree and same weight in G. Let $v_1, v_2, \ldots, v_k$ be the vertices of G', where v_i is the vertex corresponding to P_i. For each vertex v in G', we use c_v to denote the weight of the edges incident to v. It follows that $c_{v_i} = w(P_i, V \setminus P_i)$ for every vertex v_i. Also,

$$w(S_k) = \frac{\sum_{v \in V(G')} c_v}{2}\,.$$

Without loss of generality, we assume that G' is normalized, i.e., $w(S_k) = 1$.

If there exists a 2-way split in G' with the density at most $(2-\epsilon)/k$, then the theorem clearly holds. Otherwise, we assume that the density of every 2-way split is greater than $(2-\epsilon)/k$. For every vertex v in G', $S_v = (\{v\}, V(G') \setminus \{v\})$ is a 2-way split with the weight of c_v. Based on the definition of the density of splits, the density of every 2-way split is equal to its weight. Therefore, we have $d_{G'}(S_v) = c_v$. Because the density of every 2-way split is greater than $(2-\epsilon)/k$, we have $c_v > (2-\epsilon)/k$ for every vertex v.

Graph G' is a normalized, and the density of every 2-way split is at least $(2-\epsilon)/k$. Setting $\delta = 1 - \epsilon$, Lemma 3.1 implies that G' has a matching with the weight at least $\alpha = \frac{\delta^2}{16\beta(G')(1+\delta)}$.

Let $\mathcal{M}$ be the maximum matching in G'. We have $w'(\mathcal{M}) \geq \alpha$. Because G' is connected, we have

$$\beta(G') \geq 1 - 1/k \geq 2/3\,.$$

Setting $\epsilon = 1/(35\beta(G'))$, it is easy to verify that $\alpha \geq \epsilon$ for any $\beta(G') \geq 2/3$. Thus, the weight of the $\mathcal{M}$ is at least ϵ.

For every edge (a, b) in $\mathcal{M}$, let $S_{(a,b)} = (\{a\}, \{b\}, V(G') \setminus \{a, b\})$ be the 3-way split that separates a and b from all other vertices and each other. We claim that the density of at least one of these splits is at most $(2-\epsilon)/k$. For the sake of contradiction, suppose that the density of all of them is greater than $(2-\epsilon)/k$. Let U be the set of vertices which are not in $\mathcal{M}$. Recall that for every $v \in U$,

S_v is the split that separates v from all other vertices. S_v is a 2-way split, and our assumption its density is greater than $(2-\epsilon)/k$. It follows that,

$$\sum_{(a,b)\in\mathcal{M}} S_{(a,b)} + \sum_{v\in U} S_v > \frac{2(2-\epsilon)}{k}|\mathcal{M}| + \frac{(2-\epsilon)}{k}|U|\,.$$

We have $|U| = |V'| - 2|\mathcal{M}| = k - 2|\mathcal{M}|$. Therefore,

$$\sum_{(a,b)\in\mathcal{M}} S_{(a,b)} + \sum_{v\in U} S_v > 2-\epsilon\,. \tag{3}$$

Every edge which is the matching appears once in these splits, and every other edges appears twice. Recall that the weight of the matching is at least ϵ. Therefore,

$$\sum_{(a,b)\in\mathcal{M}} S_{(a,b)} + \sum_{v\in U} S_v \le 2 - w'(\mathcal{M}) \le 2-\epsilon\,,$$

which contradicts (3). Therefore, there exists an edge (a,b) in $\mathcal{M}$ such that the density of $S_{(a,b)}$ is at most $(2-\epsilon)/k$, and this completes the proof for the case G is connected with $\epsilon = 1/(35\beta(G'))$ with is a constant by Lemma 2.1.

In case G is disconnected, we construct a graph G' from G as follows:

- Add all the edges in G to G'.
- Create a new vertex u.
- For each component in G, add an edge in G' with weight ∞ from u to an arbitrary vertex in this component.

This procedure produces a connected graph G'. Moreover, every k-way split in G is also a k-way split in G', and minimum splits in G' are also minimum splits in G since weight of the new edges are ∞, and they are not in any minimum split. Clearly all the newly added edges will be in the same component of $G' - S_k$. The graph obtained by contracting all the edges whose both ends are in the same component will remain H-minor-free. Let G'' be this graph. similar to the previous case, the theorem holds for $\epsilon = 1/(35\beta(G''))$. □

Now that we know there always exists a 3-way or a 2-way split of "acceptable" ratio, we show that the density of minimum 3-way and minimum 4-way splits are also less than $2/k$ fraction of the optimal solution. The following claim is our main tool to prove this fact.

Claim 3.3. *Given an H-minor-free normalized graph $G = (V, E; w)$ with k vertices, $\delta \ge 0$ and $h < k$, let S be a h-way split such that $d_G(S) \le (1+\delta)/k$. Then, the density of a minimum $(h+1)$-way split is at most*

$$\frac{1+\delta}{k} + \frac{1-\delta}{hk}$$

Proof. Let $G' = (V', E'; w)$ be the graph obtained by removing all the edges in S from G. G' has h connected components. Also, $w(E') = 1 - w(S)$. There is a 2-way split S' in G' with the weight at most $2w(E')/(k-h+1)$. ???. Let $S'' = S \cup S'$, then we have

$$w(S'') = w(S) + w(S') = w(S)(1 - \frac{2}{k-h+1}) + \frac{2}{k-h+1}\,.$$

The weight of S is $w(S) = (h-1)d_G(S) \le (h-1)(1+\delta)/k$. Recall that $h < k$. Therefore, $1 - 2/(k-h+1) \ge 0$, and we have

$$w(S'') \le \frac{(h-1)(1+\delta)}{k}(1 - \frac{2}{k-h+1}) + \frac{2}{k-h+1}\,.$$

So,

$$w(S'') \le \frac{(h-1)(1+\delta)}{k} + \frac{2}{k-h+1}(1 - \frac{(h-1)(1+\delta)}{k})\,.$$

Thus, the density of S'' is at most

$$d_G(S'') = \frac{w(S'')}{h} \le \frac{(h-1)(1+\delta)}{hk} + \frac{2}{h(k-h+1)}(1 - \frac{(h-1)(1+\delta)}{k})\,.$$

Which is,

$$d_G(S'') \le \frac{(h-1)(1+\delta)}{hk} + \frac{2}{h(k-h+1)}(\frac{k-(h-1)(1+\delta)}{k})\,.$$

Since $k - h + 1 \ge k - (h-1)(1+\delta)$, we have,

$$d_G(S'') \le \frac{(h-1)(1+\delta)}{hk} + \frac{2}{hk} = \frac{1+\delta}{k} + \frac{1-\delta}{hk}\,. \qquad \square$$

Now we can prove that the density of minimum 3 and 4-way splits are also “acceptable”.

Claim 3.4. *Given an H-minor-free graph G and any k-way split S_k, the density of minimum 3 and 4-way splits are at most $(2-\epsilon/2)w(S_k)/k$ and $(2-\epsilon/3)w(S_k)/k$ respectively, if the separation degree of S_k is as large as those of the splits.*

Proof. Similar to the Theorem 4.2, we assume w.l.o.g. that G is connected. We contract all the edges which are not in S_k to get a new planar graph $G' = (V', E', w')$. The total weight of the edges in G' is equal to the weight of S_k. W.l.o.g., we can assume that the graph G' is normalized, i.e., $w(S_k) = w'(E') = 1$.

First, we proof our claim for a minimum 3-way split. By Theorem 3.2, there exists a split with a separation degree of at most 3 and density of at most $(2-\epsilon)/k$ in G'. Let S be this split. If the separation degree of S is 3, then our claim is proved. Otherwise, we assume that the separation degree of S is 2. Setting $\delta = 1 - \epsilon$, by Claim 3.3 the density of a minimum 3-way split is at most

$$\frac{1+\delta}{k} + \frac{1-\delta}{2k} = \frac{2-\epsilon}{k} + \frac{\epsilon}{2k} = \frac{2-\epsilon/2}{k}\,.$$

Now we consider a minimum 4-way split. We know that there exists a 3-way split with a density at most $(2-\epsilon/2)/k$. Setting $\delta = 1 - \epsilon/2$, and applying Claim 3.3, it gives us that the density of a minimum 4-way split is at most

$$\frac{1+\delta}{k} + \frac{1-\delta}{3k} = \frac{2-\epsilon/2}{k} + \frac{\epsilon/2}{3k} = \frac{2-\epsilon/3}{k}\,.$$

$\square$

Note that if Algorithm 1 picks a split in one of its steps (except the last one), it is guaranteed that no split with a lower separation degree has a lower density. We call these splits, *sparse*. Specifically, we define sparse splits as below.

Definition 3.5 (Sparse split)**.** In a graph $G = (V, E; w)$, an h-way split S is called *sparse* if for any h'-way split S' where $h' \leq h$, the following holds.

$$d_G(S) \leq d_G(S') \,.$$

The following theorem shows that combining some low-density sparse splits, results a low-density split.

Theorem 3.6. *Let $G = (V, E; w)$ be an H-minor-free graph, S be a k-way split in G, and $a_1, a_2, \ldots a_l$ be integers such that $\sum_{i=1}^{l} a_i < k$. Let $C_1, C_2, \ldots, C_l$ be l splits where C_i is a minimum $(a_i + 1)$-way split in $G_i = G - \bigcup_{j=1}^{i-1} C_j$. Let $S_i = S \setminus \bigcup_{j=1}^{i-1} C_j$ be a b_i-way split in G_i for every $1 \leq i \leq l$. Given a $\delta \geq 0$, suppose that for every C_i, we have*

$$d_G(C_i) \leq \frac{(1+\delta) w(S_i)}{b_i} \,.$$

Also, suppose that every split except the last one is sparse, i.e., C_i is sparse in G_i for every $i < l$. Then,

$$d_G(\bigcup_{i=1}^{l} C_i) \leq \frac{(1+\delta) w(S)}{k} \,.$$

Proof. We prove this theorem by induction on l. When $l = 1$, the density of C_1 is at most $(1+\delta)w(S_1)/k$. Since $S_1 = S$, the theorem holds. For the induction step suppose that $l \geq 2$, and the theorem holds for any $l - 1$ splits. By induction hypothesis, for the last $l - 1$ splits we have

$$d_G(\bigcup_{i=2}^{l} C_i) \leq \frac{(1+\delta) w(S_2)}{b_2} \,,$$

since S_2 is a b_2-way split in G_2. It implies that

$$w(\bigcup_{i=2}^{l} C_i) \leq \frac{(1+\delta) w(S_2)}{b_2} \sum_{i=2}^{l} a_i \,.$$

Since C_1 is a $(a_1 + 1)$-split in G and $S_2 = S \setminus C_1$, we have $k - a_1 \leq b_2 \leq k$. Let $S' = S \cap C_1$. It follows that S' is a $(k - b_2 + 1)$-way split in G. Also, we have $1 \leq k - b_2 + 1 \leq a_1 + 1$. We prove the induction by considering two cases on $k - b_2 + 1$.

- If $k - b_2 + 1 = 1$, then $b_2 = k$ and S_2 is a k-way split in G_2. Therefore,

$$\begin{aligned} w(\bigcup_{i=1}^{l} C_i) &\leq w(C_1) + \frac{(1+\delta) w(S_2)}{k} \sum_{i=2}^{l} a_i \\ &\leq \frac{(1+\delta) w(S)}{k} a_1 + \frac{(1+\delta) w(S_2)}{k} \sum_{i=2}^{l} a_i \\ &\leq \frac{(1+\delta) w(S)}{k} \sum_{i=1}^{l} a_i \,. \end{aligned}$$

  It implies that

$$d_G(\bigcup_{i=1}^{l} C_i) = \frac{w(\bigcup_{i=1}^{l} C_i)}{\sum_{i=1}^{l} a_i} \leq \frac{(1+\delta) w(S)}{k} \,.$$

  This completes the induction step for this case.

- Otherwise, $k - b_2 + 1 > 1$, i.e., the separation degree of $S' = S \cap C_1$ is at least 2. By sparsity of the split C_1, we have

$$d_G(C_1) \le d_G(S') \Rightarrow \frac{w(C_1)}{a_1} \le \frac{w(S')}{k - b_2}\,.$$

Therefore,

$$w(S') \ge \frac{w(C_1)(k - b_2)}{a_1}\,. \tag{4}$$

It derives that, the weight of the union of $C_1, C_2, \ldots, C_l$ is

$$w(\bigcup_{i=1}^{l} C_i) \le w(C_1) + \frac{(1+\delta)w(S_2)}{b_2} \sum_{i=2}^{l} a_i\,.$$

Since $S_2 = S - S'$, we have $w(S_2) = w(S) - w(S')$. Therefore,

$$w(\bigcup_{i=1}^{l} C_i) \le w(C_1) + \frac{(1+\delta)(w(S) - w(S'))}{b_2} \sum_{i=2}^{l} a_i\,.$$

By (4), we have

$$w(\bigcup_{i=1}^{l} C_i) \le w(C_1) + \frac{(1+\delta)(w(S) - w(C_1)(k - b_2)/a_1)}{b_2} \sum_{i=2}^{l} a_i\,.$$

Let $a = \sum_{i=1}^{l} a_i$. We claim that the weight of the split $\bigcup_{i=1}^{l} C_i$ is at most $a(1+\delta)w(S)/k$. Define the function g as

$$g(x) = x + \frac{(1+\delta)(w(S) - x(k - b_2)/a_1)}{b_2} \sum_{i=2}^{l} a_i\,,$$

which is equal to

$$g(x) = x + \frac{(1+\delta)(w(S) - x(k - b_2)/a_1)(a - a_1)}{b_2}\,.$$

Then,

$$w(\bigcup_{i=1}^{l} C_i) \le g(w(C_1))\,.$$

Since g is linear in x, it is sufficient to show that our claim holds for both ends of g.

  ◇ For $g(0)$ we have

  $$g(0) = \frac{(1+\delta)w(S)(a - a_1)}{b_2}\,.$$

  Since $b_2 \ge k - a_1$, we have

  $$g(0) \le \frac{(1+\delta)w(S)(a - a_1)}{k - a_1}\,.$$

  It is easy to verify that $(a - a_1)/(k - a_1) \le a/k$ for every $a \le k$. Therefore,

  $$g(0) \le \frac{(1+\delta)w(S)a}{k}\,,$$

  which proves our claim.

◇ For $g(a_1(1+\delta)w(S)/k)$ we have

$$g(a_1(1+\delta)w(S)/k) = \frac{a_1(1+\delta)w(S)}{k} + \frac{(1+\delta)(w(S)-(1+\delta)w(S)(k-b_2)/k)(a-a_1)}{b_2}$$
$$= (1+\delta)w(S)(\frac{a_1}{k} + \frac{(1-(1+\delta)(k-b_2)/k)(a-a_1)}{b_2}).$$

$1+\delta \geq 1$. Therefore,

$$g(a_1(1+\delta)w(S)/k) \leq (1+\delta)w(S)(\frac{a_1}{k} + \frac{(1-(k-b_2)/k)(a-a_1)}{b_2})$$
$$= (1+\delta)w(S)(\frac{a_1}{k} + \frac{(b_2/k)(a-a_1)}{b_2}).$$

Simplifying the inequality gives that

$$g(a_1(1+\delta)w(S)/k) \leq (1+\delta)w(S)(\frac{a_1}{k} + \frac{a-a_1}{k}) = \frac{(1+\delta)w(S)a}{k}.$$

Thus, $w(\bigcup_{i=1}^{l} C_i) \leq a(1+\delta)w(S)/k$. It follows that $d_G(\bigcup_{i=1}^{l} C_i) \leq (1+\delta)w(S)/k$.

We proved the induction step for both cases, and it completes the proof for our theorem. □

Finally we can establish the approximation guarantee of the greedy algorithm.

Theorem 3.7. *The approximation ratio of Algorithm 1 is $2-\epsilon/3$ in minor-free graphs.*

Proof. If $k \leq 4$, the algorithm finds the minimum k-way split at its only step. Therefore, the weight of the split returned by the algorithm is the optimal solution. Otherwise, we suppose that $k > 4$.

Let S_{OPT} be a minimum k-way split. The algorithm successively finds a split with the separation degree at most 3 that has a minimum density. The only exception is the last split that it picks which is either minimum 3 or 4-way split.

Let $C_1, C_2, \ldots, C_l$ be the splits picked by the algorithm, $G_i = G - \bigcup_{j=1}^{i-1} C_j$, and $S_i = S_{\mathsf{OPT}} - \bigcup_{j=1}^{i-1} C_j$ be a b_i-way split in G_i. By Theorem 3.2, $d_G(C_i) \leq (2-\epsilon)w(S_i)/b_i$ for every $i < l$. Also by Claim 3.4, $d_G(C_l) \leq (2-\epsilon/3)w(S_l)/b_l$. Also, all the splits $C_1, C_2, \ldots, C_{l-1}$ are sparse. Let $C = \bigcup_{i=1}^{l} C_i$ be the k-cut returned by the algorithm. It follows from Theorem 3.6 that

$$d_G(C) \leq (2-\epsilon/3)w(S_{\mathsf{OPT}})/k.$$

Therefore,

$$w(C) \leq (2-\epsilon/3)w(S_{\mathsf{OPT}}).$$

□

Corollary 3.8. *The approximation ratio of Algorithm 1 is 1.9968... in planar graphs.*

Proof. The ϵ derived from Theorem 3.2 is $1/(35\beta(G'))$ where G' is a minor of G. If G is a planar graph, then G' is also planar. Therefore, $\beta(G') \leq 3$, and Theorem 3.2 holds for $\epsilon = 1/(35 \cdot 3) = 1/105$. Hence, the approximation ratio of Algorithm 1 in planar graphs is $2-\epsilon/3 = 2-1/315$ which is 1.9968.... □

Putting together Theorem 3.7 with the bounds established for ϵ in this section yields Theorem 1.1: there exists a polynomial-time algorithm for k-cut whose approximation factor for minor-free graphs is a constant factor smaller than 2. The approximation guarantee is 1.9968... in planar graphs.

Algorithm 2: PTAS for the k-cut Problem in Minor-free Graphs

Data: A planar graph G, integer k, and $\epsilon > 0$

$C = \emptyset$.
while separation degree of C is less than k **do**
    Let $G' = G - C$ be the graph obtained by removing all the previous cuts from G.
    Let d be the separation degree of C.
    if $k - d \leq h(\epsilon)(2 + 1/\epsilon)$ **then**
        Let C' be a minimum $(k - d + 1)$-way split in G'.
    else
        Let C' be a split which its density is minimum among all splits with the separation degree at most $h(\epsilon)$.
    $C = C \cup C'$.
return C.

4 Polynomial Time Approximation Scheme

In this section we generalize our method to derive a polynomial time approximation scheme (PTAS) for the k-cut problem in minor-free graphs. Recall that in the last section we showed that approximation ratio of a natural greedy algorithm which successively removes the lowest density split with the separation degree of at most 3 is less than 2 in minor-free graphs. Our main observation for proving this bound was to show that the density of this split is at most $(2 - \epsilon)/k$ fraction of the weight of a minimum k-way cut.

We generalize our method, and provide a PTAS for the k-cut problem in minor-free graphs. In this section we show that the density of minimum weighted splits converges to OPT/k if we consider splits with a larger separation degree where OPT is the weight of the optimal solution. For an $\epsilon > 0$, we first show that there exists a constant $h(\epsilon)$ such that there exists a split with the separation degree of at most $h(\epsilon)$ and the density of at most $(1+\epsilon)\mathsf{OPT}/k$. To this purpose, we use the separation theorem which shows that in every minor-free graph with n vertices, the removal of $O(\sqrt{n})$ vertices, can partition the graph into two parts such that each of them has at most $2n/3$ vertices.

Theorem 4.1 (Alon et al. [1] , Lipton and Tarjan [19])**.** *Let G be an H-minor-free graph with n vertices, then there exists a separator of size of at most $c_1\sqrt{n}$ such that c_1 is a constant only depending on $|V(H)|$, and removal of this separator partitions the graphs into two parts each of which has at most $2n/3$ vertices.*

The following theorem, is our main observation to derive a PTAS for the k-cut problem.

Theorem 4.2. *Given an H-minor-free $G = (V, E; w)$ and $\epsilon > 0$ there exists a constant $h(\epsilon)$ such that for any $k \geq h(\epsilon)$ and k-way split S_k, there exists a split with the separation degree of at most $h(\epsilon)$ and the density of at most $(1 + \epsilon)w'(S_k)/k$.*

Proof. Similar to Theorem 3.2, we assume w.l.o.g. that G is connected. We contract all the edges which are not in S_k to get a new minor-free graph $G' = (V', E'; w')$. The total weight of the edges in G' is equal to the weight of S_k. W.l.o.g., we can assume that the graph G' is normalized, i.e., $w(S_k) = 1$.

The following lemma is directly derived from Theorem 4.1 which shows that for any $\delta > 0$, there exists $O(k\delta)$ vertices such that removal of them partitions G' into parts with the size at most $1/\delta^2$.

Lemma 4.3. *For any H-minor-free graph G with n vertices, there exists a constant c_2 such that for any $\delta > 0$, removal of $c_2 n\delta$ vertices of G partitions it into parts with the size at most $1/\delta^2$.*

Proof. The proof is almost alike to the proof of the similar lemma in [11]. We recursively find and remove the separator of Theorem 4.1 in each part until its size become at most $1/\delta^2$. Let $b(n)$ be the number of vertices removed in an H-minor-free graph with n vertices. The removal of the separator in Theorem 4.1 partitions the graph into two parts such that each of them has at least $n/3$ vertices. Let $n\alpha$ be the size of the first part, then the size of the other part is at most $n(1-\alpha)$. Therefore, we have

$$b(n) \leq c_1\sqrt{n} + b(n\alpha) + b(n(1-\alpha))\,,$$

where $1/3 \leq \alpha \leq 2/3$. Also, we have

$$b(n) = 0\,,$$

for any $n \leq 1/\delta^2$. It can be shown by induction that

$$b(n) \leq c_2 n\delta - d\sqrt{n}\,,$$

for some constants c_2 and d. □

Note that G' is H-minor-free. Therefore, there exists a constant c_2 such that Lemma 4.3 holds for G'. Let $\delta = \epsilon/(c_2(1+\epsilon))$, by Lemma 4.3, there is a separator of size at most $c_2 k\delta$ that removing it partitions G' it into several parts such that size of each of them is at most $1/\delta^2$.

Let $P_1, P_2, \cdots, P_l$ be these parts where P_i is the set of vertices in the part i. Let $P_i = \{v_{i,1}, v_{i,2}, \cdots\}$, and C_i be the split that separates each vertex in P_i from every other vertex in G', i.e., $C_i = (\{v_{i,1}\}, \{v_{i,2}\}, \cdots, V' \setminus P_i)$. We claim that the density of at least one of C_i is at most $(1+\epsilon)/k$. For the sake of the contradiction, suppose that the density of every C_i is greater than $(1+\epsilon)/k$.

Note that every edge in the splits C_i, is either between two vertices in a same part, or between a vertex of the separator and another vertex. Therefore, each edge appears at most once in these splits. Thus,

$$\sum_{i=1}^{l} w'(C_i) \leq 1\,. \tag{5}$$

On the other hand, the density of every C_i is greater than $(1+\epsilon)/k$. Therefore, we have

$$\sum_{i=1}^{l} w'(C_i) > \frac{1+\epsilon}{k}\sum_{i=1}^{l} |P_i|\,.$$

Since the size of the separator is at most $c_2 k\delta$, we have,

$$\begin{aligned}\sum_{i=1}^{l} w'(C_i) &> \frac{1+\epsilon}{k}\sum_{i=1}^{l} |P_i| \\ &\geq \frac{1+\epsilon}{k} k(1-c_2\delta) \\ &= (1+\epsilon)(1-c_2\delta)\,.\end{aligned}$$

Substituting δ with $\epsilon/(c_2(1+\epsilon))$, gives us,

$$\sum_{i=1}^{l} w'(C_i) > (1+\epsilon)(1-c_2\delta) = (1+\epsilon)(1-\frac{\epsilon}{1+\epsilon}) = 1 \tag{6}$$

Inequality (5) contradicts (6). Therefore, for at least one of the C_i, its density is at most $(1+\epsilon)/k$. Since size of the each P_i is at most $1/\delta^2$, the separation degree of our splits is at most $1/\delta^2+1$ which is a constant. □

To complete the proof we show that the density of the split which Algorithm 2 picks in its last step is not very large. Note that the separation degree of the split which the algorithm finds in the last step is either k or it is at least $h(\epsilon)/(\epsilon+1)$.

Lemma 4.4. *Given a minor-free graph G and integers $s \geq h(\epsilon)(1+1/\epsilon)$ and $k \geq s$, let S be a minimum s-way split in G. Then, for any k-way split S_k, we have*

$$d_G(S) \leq \frac{(1+2\epsilon)w(S_k)}{k}.$$

Proof. Similar to the Theorem 4.2, we assume w.l.o.g. that G is connected. We contract all the edges which are not in S_k to get a new planar graph $G' = (V', E'; w')$. The total weight of the edges in G' is equal to the weight of S_k. W.l.o.g., we can assume that the graph G' is normalized, i.e., $w(S_k) = w'(E') = 1$.

Now we want to remove some edges in G' to increase the number of components in G' by $s-1$. While the number of components in G' is at most $s-h(\epsilon)$, we find a split with the separation degree at most $h(\epsilon)$ that has the minimum density. Let $C_1, C_2, \ldots, C_l$ be these splits and $(a_1+1), (a_2+1), \ldots, (a_l+1)$ be their size respectively. Let $G'_i = G' - \bigcup_{j=1}^{i-1} C_i$ for every C_i. It follows that G'_i is a $(k - \sum_{j=1}^{i-1} a_j)$-way split in G'. According to Theorem 4.2 we have

$$d_{G'}(C_i) \leq \frac{(1+\epsilon)w'(G'_i)}{k - \sum_{j=1}^{i-1} a_j}.$$

Let $C = \bigcup_{i=1}^{l} C_i$, be a s'-way split for G' where $s' = 1 + \sum_{i=1}^{l} a_i$. Since the number of connected components in $G' - C$ is greater than $s - h(\epsilon)$, we have $s' > s - h(\epsilon)$. Therefore, $s' > s - h(\epsilon) \geq h(\epsilon)/\epsilon$. Also, $s - s' < h(\epsilon)$. By applying Theorem 3.6 to the splits $C_1, C_2, \ldots, C_l$, we have

$$d_{G'}(C) \leq \frac{(1+\epsilon)w'(G')}{k} = \frac{1+\epsilon}{k}.$$

Therefore,

$$w'(C) \leq \frac{(1+\epsilon)(s'-1)}{k}.$$

If $s = s'$, we have found a split with the density of at most $(1+\epsilon)/k$ and proved the theorem. Otherwise, we can assume that $s > s'$. By setting $\delta = \epsilon$, Claim 3.3 implies that the density of a minimum $(s'+1)$-way split is at most

$$\frac{1+\epsilon}{k} + \frac{1-\epsilon}{s'k} < \frac{1+\epsilon}{k} + \frac{1}{s'k}.$$

Since s' is larger than $h(\epsilon)/\epsilon$, the density of a minimum $(s'+1)$-way split is at most

$$\frac{1+\epsilon}{k} + \frac{1}{s'k} < \frac{1+\epsilon}{k} + \frac{\epsilon}{h(\epsilon)k} = \frac{1+\epsilon(1+1/h(\epsilon))}{k}.$$

Repeatedly applying 3.3 implies that for any $a > 0$, the density of a minimum $(s'+a)$-way split is at most

$$\frac{1+\epsilon(1+a/h(\epsilon))}{k}.$$

Therefore, the density of a minimum s-way split is at most

$$\frac{1+\epsilon(1+(s-s')/h(\epsilon))}{k} < \frac{1+\epsilon(1+h(\epsilon)/h(\epsilon))}{k} = \frac{1+2\epsilon}{k}\,.$$

□

Now we are ready to prove that Algorithm 2 is a PTAS for the k-way cut in minor-free graphs.

Theorem 4.5. *Given an $\epsilon > 0$, the approximation ratio of Algorithm 2 is $1+2\epsilon$ in minor-free graphs.*

Proof. The analysis is very similar to the Theorem 3.7. If $k \leq h(\epsilon)(2+1/\epsilon)+1$, the algorithm finds the minimum k-way split at its only step. Therefore, the weight of the split returned by the algorithm is the optimal solution. Otherwise, we suppose that $k > h(\epsilon)(2+1/\epsilon)+1$.

Let S_{OPT} be a minimum k-way split. The algorithm successively finds a split with the separation degree of at most $h(\epsilon)$ that has a minimum density. The only exception is the last split that it picks that is a split with the separation degree at least $h(\epsilon)(1+1/\epsilon)$.

Let $C_1, C_2, \ldots, C_l$ be the splits picked by the algorithm, $G_i = G - \bigcup_{j=1}^{i-1} C_j$, and $S_i = S_{\mathsf{OPT}} - \bigcup_{j=1}^{i-1} C_j$ be a b_i-way split in G_i. By Theorem 4.2, $d_G(C_i) \leq (1+\epsilon)w(S_i)/b_i$ for every $i < l$. Also by Lemma 4.4, $d_G(C_l) \leq (1+2\epsilon)w(S_l)/b_l$. Also, all the splits $C_1, C_2, \ldots, C_{l-1}$ are sparse. Let $C = \bigcup_{i=1}^{l} C_i$ be the k-cut returned by the algorithm. It follows from Theorem 3.6 that

$$d_G(C) \leq (1+2\epsilon)w(S_{\mathsf{OPT}})/k\,.$$

Therefore,

$$w(C) \leq (1+2\epsilon)w(S_{\mathsf{OPT}})\,.$$

□

Theorem 4.5 establishes our second main result.

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